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FLAT EMBEDDINGS OF NOETHERIAN ALGEBRAS IN ARTINIAN RINGS

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ABSTRACT

It is shown that a noetherian algebra R with finite Gelfand–Kirillov dimension and right primary decomposition can be embedded in an artinian ring S, and that S is flat as a left R-module if and only if all right associated primes are minimal. If R is irreducible then such a flat embedding is possible if and only if R has an artinian quotient ring. Also, the existence of a left flat embedding in an artinian ring allows an explicit description of the prime middle annihilators of R.

1. Introduction

The question whether all (right and left) noetherian rings can be embedded in artinian rings has only recently been answered in the negative. In [5], Dean and Stafford have shown that a certain factor ring of $U(\mathfrak{sl}(2,\mathbb{C}))$, the enveloping algebra of the smallest simple complex Lie algebra, cannot be so embedded. In a subsequent paper [4], Dean showed that indeed the enveloping algebra of a complex Lie algebra $\mathfrak{g}$ has a nonembeddable factor ring if and only if $\mathfrak{g}$ is not solvable. Since by a result of Brown and Lenagan [2], a complex Lie algebra $\mathfrak{g}$ is solvable if and only if all factor rings of $U(\mathfrak{g})$ have a primary decomposition, this clarifies, at least for the enveloping algebras, the connection between primary decomposition and embeddability in artinian rings. We show that any noetherian algebra with finite Gelfand–Kirillov dimension that has a primary decomposition

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can be embedded in an artinian ring. The base for this result and others presented in this paper is a remarkable theorem due to Schofield [18], which asserts that an algebra R over a field k is embeddable in an artinian ring if and only if R admits a faithful Sylvester rank function λ. The non-embeddability of Dean's examples is established by showing that no such function exists for the rings in question. On the positive side, Blair and Small [1] have shown that the function $\lambda(M) = \rho(M)/\rho(R_R)$ for finitely generated right modules over a right noetherian k-algebra R, where ρ denotes the reduced rank, is a Sylvester rank function that is faithful whenever the ideal $K = \{a \in R \mid \rho(aR) = 0\}$ is zero. In this case, R embeds in a simple artinian ring S such that ${}_RS$ is flat. Starting from this, we show that the existence of such a flat embedding is actually equivalent to $K = 0$, and that, for a noetherian k-algebra R, this is the case if and only if $\mathrm{Ass}(R_R)$ consists of minimal primes, so that the existence of a flat artinian embedding in the commutative case is equivalent to the existence of an artinian quotient ring. Furthermore, a noetherian k-algebra R has a left flat artinian embedding if and only if R is a finite subdirect product of irreducible right P_i-primary rings R_i such that each P_i is a minimal prime ideal of R. If R has also finite Gelfand–Kirillov dimension, then each of the components R_i has in fact an artinian quotient ring $Q(R_i)$, although R may fail to have one, and indeed, the embedding of R into the artinian ring $\bigoplus Q(R_i)$ need not be the flat embedding that exists for R.

This paper ends with an explicit description of the set of prime middle annihilators of a noetherian k-algebra with finite Gelfand–Kirillov dimension that has a left flat embedding in an artinian ring. It turns out that this set coincides with the set of all those prime ideals that are minimal over a left annihilator ideal.

2. Definitions, Notations, and Some Technical Results

Throughout, k denotes a field, and R is a k-algebra, occasionally just a ring with unit element, modules are generally unitary right R-modules. Most of our notation is standard, what follows is a list of the most frequently used terms. For details, the reader is referred to the book by McConnell and Robson [17].

$$\begin{aligned}
\operatorname{minspec}(R) &= \text{ set of minimal prime ideals of } R. \\
N = N(R) &= \text{ prime radical of } R = \bigcap\{P \mid P \in \operatorname{minspec}(R)\}. \\
r(X) &= \text{ right annihilator of the set } X. \\
\ell(X) &= \text{ left annihilator of the set } X. \\
{}'\mathcal{C}(I) &= \text{ set of elements that are left regular modulo the ideal } I \\
&= \{c \in R \mid xc \in I \rightarrow x \in I\}. \\
\mathcal{C}'(I) &= \text{ set of elements that are right regular modulo the ideal } I. \\
\mathcal{C}(I) &= {}'\mathcal{C}(I) \cap \mathcal{C}'(I). \\
Z(M) &= \text{ singular submodule of the module } M \\
&= \{m \in M \mid r(m) \text{ essential in } R_R\}. \\
\rho(M) &= \text{ reduced rank of the module } M.
\end{aligned}$$

A prime ideal P of R is **associated** with the right R-module M if there exists a nonzero submodule N of M such that $P = r(N')$ for all $0 \neq N' \subseteq N$. The module M is P-**primary** if P is its only associated prime.

$$\operatorname{Ass}(M) = \text{ set of all primes associated with the module } M.$$

An ideal I of R and the ring R/I are **right** P-**primary** if R/I, considered as a right module, is P-primary. The ring R has **right primary decomposition** if there exist finitely many right primary ideals intersecting in zero. An ideal M of R is a **middle annihilator** if there exist ideals A and B with $AB \neq 0$ such that $M = \operatorname{Mid}(A, B) = \{x \in R \mid AxB = 0\}$.

A **Sylvester module rank function**, more briefly a **Sylvester rank function,** on the ring R is a function λ from finitely presented right R-modules to nonnegative numbers in $(1/n)\mathbb{Z}$ for some fixed positive integer n, such that

(i) $\lambda(R) = 1$.

(ii) $\lambda(M \oplus N) = \lambda(M) + \lambda(N)$.

(iii) If $L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of finitely presented right R-modules, then $\lambda(N) \leq \lambda(M) \leq \lambda(L) + \lambda(N)$.

If (ii) and (iii) are replaced by the stronger condition that $\lambda(M) = \lambda(L) + \lambda(N)$ whenever $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of finitely generated right R-modules, then the Sylvester module rank function λ is called **exact**, and it is said to be **faithful** if $\lambda(R/A) < 1$ for each right ideal $A \neq 0$ of R.

2.1 THEOREM (Schofield [18]): *A Sylvester rank function λ on a k-algebra R taking values in $(1/n)\mathbb{Z}$ arises from homomorphism $f : R \to M_n(D) = S$, where D is a division ring, with* $\ker(f) = \{a \in R \mid \lambda(R/aR) = 1\}$. *The simple artinian ring S is flat as a left R-module if and only if λ is exact.*

Since a right artinian k-algebra R can be embedded in a simple artinian ring S such that ${}_RS$ is flat, using the exact Sylvester rank function on R that arises from the composition length, it is clear that whenever a k-algebra R is embeddable in an artinian k-algebra S such that ${}_RS$ is flat, then one may as well assume that S is simple artinian. We say, for short, that R has a **left flat artinian embedding.**

Frequent use is made of the **Gelfand–Kirillov dimension**, or **GK-dimension** for short, of k-algebras and their modules. Whenever one has to distinguish between the right and left hand side, we indicate this by a subscript. Thus, for the S-R-bimodule $M, \mathrm{GK}_S(M)$ and $\mathrm{GK}(M)_R$ denote its GK-dimensions as a left S-module and right R-module, respectively. For basic properties concerning GK-dimension, we refer to [13] or Chapter 8 of [17]. Note that we do not assume GK-dimension to be exact, that is, we do not take for granted that $\mathrm{GK}(M) = \max\{\mathrm{GK}(S), \mathrm{GK}(M/S)\}$ whenever S is a submodule of M. We also do not assume that $\mathrm{GK}(R) = \mathrm{GK}(R/N)$.

Definition: The k-algebra R is **right (left) GK-homogeneous** if $\mathrm{GK}(A) = \mathrm{GK}(R)$ for each right (left) ideal $A \neq 0$, it is **right (left) weakly GK-homogeneous** if $\mathrm{GK}(A) \geq \mathrm{GK}(R/N)$ for each right (left) ideal $A \neq 0$.

2.2 LEMMA: *The following are equivalent for the right noetherian k-algebra R.*
(1) *R is right weakly* GK-*homogeneous and* $\mathrm{GK}(R) = \mathrm{GK}(R/N)$.
(2) *R is right* GK-*homogeneous.*

Proof: If R is right GK-homogeneous, then

$$\mathrm{GK}(R) = \mathrm{GK}(\ell(N))_R \leq \mathrm{GK}(R/N) \leq \mathrm{GK}(R).$$

The rest is trivial.

2.3 PROPOSITION: *Let R be a right noetherian k-algebra. Then*
(a) $\mathrm{GK}(R/N) \leq \mathrm{GK}(E)$ *for every essential right ideal E of R.*
(b) $\mathrm{GK}(R/N) = \mathrm{GK}(R/P)$ *for some prime ideal* $P \in \mathrm{Ass}(R_R)$.

Proof: (a) Note that, since R is right noetherian, we have that $\mathrm{GK}(RE)_R = \mathrm{GK}(E)$, as RE is a finite sum of right ideals of the form $rE, r \in R$, each being a homomorphic image of E_R. Note also that RE is finitely generated on the right, so that $\mathrm{GK}(R/\ell(RE)) = \mathrm{GK}_R(RE) \leq \mathrm{GK}(RE)_R$ by [13, Lemma 5.3]. Since RE is an essential right ideal, $\ell(RE) \subseteq Z(R_R) \subseteq N$, so $\mathrm{GK}(R/N) \leq \mathrm{GK}(R/\ell(RE))$.

(b) Let $E = \ell(D)$, where $D = \cap\{P \mid P \in \mathrm{Ass}(R_R)\}$, so E is an essential right ideal. Since E is a right R/D-module, and since $N \subseteq D$, it follows from (a) that

$$\mathrm{GK}(R/N) \leq \mathrm{GK}(E)_R \leq \mathrm{GK}(R/D) \leq \mathrm{GK}(R/N),$$

so $\mathrm{GK}(R/N) = \max\{\mathrm{GK}(R/P) \mid P \in \mathrm{Ass}(R_R)\}$.

Note that if R is a right noetherian k-algebra with $\mathrm{GK}(R) < \infty$, then any prime ideal P with $\mathrm{GK}(R/N) = \mathrm{GK}(R/P)$ must be a minimal prime. Thus we have

2.4 COROLLARY: *Let R be a right noetherian, right P-primary k-algebra with $\mathrm{GK}(R) < \infty$. Then P is a minimal prime ideal.*

Note that this is no longer true in general when R has infinite GK-dimension; see Example 3.6 below.

Several of our results are true for both noetherian rings and right noetherian right fully bounded rings, right FBN-rings for short. It turns out that what is used in the proofs is that every right ideal A is **finitely annihilated**, that is, $r(A) = \bigcap_{i=1}^{n} r(a_i)$ for a finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq A$.

3. Noetherian Algebras with Left Flat Artinian Embeddings

In [1], Blair and Small showed that a right noetherian right Krull-homogeneous k-algebra R embeds in a simple artinian ring S such that ${}_RS$ is flat. This carries over to the weakly GK-homogeneous case, virtually the same proof can be used, although we present a shorter one.

3.1 THEOREM: *Let R be a right noetherian right weakly GK-homogeneous K-algebra with $GK(R) < \infty$. Then R can be embedded in a simple artinian ring S such that ${}_RS$ is flat.*

Proof: We show that the ideal $K = \{a \in R \mid \rho(aR) = 0\}$ is zero, the claim follows from this by Theorem 1 of [1]. Assume that $K \neq 0$. Since $\ell(N)$ is an essential right ideal, $K \cap \ell(N) \neq 0$, so let $0 \neq y \in K \cap \ell(N)$, and let $c \in r(y) \cap \mathcal{C}(N)$. Then $r(y) \supseteq cR + N$, so

$$\mathrm{GK}(yR) = \mathrm{GK}(R/r(y)) \leq \mathrm{GK}(R/cR + N) < \mathrm{GK}(R/N),$$

contradicting the weak GK-homogeneity of R. ∎

It follows from this result that a right noetherian right weakly GK-homogeneous k-algebra with finite GK-dimension satisfies the descending chain condition for right annihilators. The following proposition shows that in the right primary case the converse is also true.

3.2 PROPOSITION: *The following are equivalent for a right noetherian right primary k-algebra with finite GK-dimension.*

(1) *Each right ideal of R is finitely annihilated.*

(2) *R is right weakly GK-homogeneous.*

(3) *R satisfies the descending chain condition for right annihilators.*

Proof: (1) → (2): Let $\mathrm{Ass}(R_R) = P$. By Proposition 2.3(b), $\mathrm{GK}(R/N) = \mathrm{GK}(R/P)$. Let A be a nonzero right ideal, and let $U \neq 0$ be a uniform right ideal, $U \subseteq A$, such that $r(U) = P$. Since U is finitely annihilated, U is not $\mathcal{C}(P)$-torsion, so U must be torsionfree and hence isomorphic to a uniform right ideal of R/P. Consequently, $\mathrm{GK}(A) \geq \mathrm{GK}(U) = \mathrm{GK}(R/P) = \mathrm{GK}(R/N)$.

(2) → (3): This is clear since R embeds in an artinian ring by Theorem 3.1.

(3) → (1): This is trivial. ∎

Since a right FBN-ring has right primary decomposition by a result of Gordon [8, Corollary 2.4], the following corollary generalizes Theorem 4 of Blair and Small [1] and answers, at least for right FBN-algebras with finite GK-dimension, their question posed after the proof of that result.

3.3 COROLLARY: *A right noetherian, right fully bounded k-algebra with finite GK-dimension can be embedded in an artinian ring.*

In [15, Corollary 4.2], Lenagan has shown that any factor ring of the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional solvable complex Lie algebra $\mathfrak{g}$ can be embedded in an artinian ring. Since any such ring has a primary decomposition by [2], the following corollary generalizes this result.

3.4 COROLLARY: *Let R be a noetherian k-algebra with finite GK-dimension. If R has right primary decomposition, then R can be embedded in an artinian ring.*

Next, we show that for a right noetherian k-algebra whose right ideals are finitely annihilated, Theorem 1 of Blair and Small [1] has a converse.

3.5 THEOREM: *Let R be a right noetherian ring whose right ideals are finitely annihilated. Then the following are equivalent.*

(1) $\mathrm{Ass}(R_R) \subseteq \mathrm{minspec}(R)$.

(2) *There exist irreducible right P_j-primary ideals $I_j, j = 1, \ldots, n$ with*

$$I_1 \cap I_2 \cap \ldots \cap I_n = 0 \quad \text{and} \quad \{P_1, \ldots, P_n\} \subseteq \mathrm{minspec}\,(R).$$

(3) $K = \{a \in R \mid \rho(aR) = 0\} = 0$.

If, furthermore, R is a k-algebra, then the above are equivalent to

(4) *R embeds in a simple artinian ring S such that ${}_RS$ is flat.*

Proof: (1) $\rightarrow$ (2): Pick irreducible ideals $I_1, \ldots, I_n$ such that their intersection is zero yet $\bigcap_{j \neq i} I_j \cap X_i \neq 0$ whenever X_i is an ideal that properly contains I_i. This is possible in any ring with ascending chain condition for ideals. Now let P_j/I_j be the unique maximal right associated prime of $R/I_j, j = 1, \ldots, n$. Then $X_jP_j \subseteq I_j$ for some ideal $X_j \supsetneqq I_j$, so $X_j \cap \bigcap_{i \neq j} I_i \neq 0$ and $(X_j \cap \bigcap_{i \neq j} I_i)P_j = 0$. Thus $P_j \subseteq r(X_j \cap \bigcap_{i \neq j} I_i) \subseteq P$ for some $P \in \mathrm{Ass}(R_R)$. Since P is assumed to be minimal, $P_j = P$ follows. The rest is now clear.

(2) $\rightarrow$ (3): Assume that $K \neq 0$, so $H = K \cap \ell(N) \neq 0$. Since H is finitely annihilated, $Hc = 0$ for some $c \in \mathcal{C}(N)$. Now, H is not contained in each $I_j, j = 1, \ldots, n$, so $H \not\subseteq I_1$, say. Since $c \in r(H + I_1/I_1) \subseteq P_1 = \mathrm{Ass}(R/I_1)$, we get a contradiction, since P_1 is assumed to be a minimal prime.

(3) $\rightarrow$ (1): Let $P \in \mathrm{Ass}(R_R)$. If P were not minimal, then there would be an element $c \in P \cap \mathcal{C}(N)$. Now $\ell(P)c = 0$, so $\ell(P) \subseteq K = 0$, and P would not be right associated with R.

Let now R be a k-algebra. Then (4) follows from (3) by Theorem 1 of [1].

Conversely, assume that there is an embedding $f : R \rightarrow S = M_n(D), D$ a division ring, such that ${}_R(S)$ is flat. The corresponding Sylvester rank function λ from finitely generated right R-modules into $(1/n)\mathbb{Z}$ is exact, so $n\lambda$ is an additive rank function in the sense of [12]. Now let $x \in K$, so $\rho(xR) = 0$. By [12, Lemma 1.3], $n\lambda(xR) = 0$, so $\lambda(xR) = 0$. Thus $\lambda(R/xR) = \lambda(R) = 1$, so $x \in \ker(f) = 0$. Hence $K = 0$, as claimed. ∎

Remark: The implications (1) $\to$ (2) and (3) $\to$ (1) hold for any right noetherian ring. Note also that (3) and (4) are equivalent if R is merely a right noetherian k-algebra. Thus, a right noetherian k-algebra whose right associated primes are not all minimal cannot possibly have a left flat artinian embedding. In this context it is perhaps worthwhile recalling the following example due to Blair and Small [1], as it illustrates the preceding results. In order to make this note reasonably self-contained, we present the construction.

3.6 Example: [1, p.17] Let $A = \mathbb{R}[x]_{(x^2+1)}$, the ring of polynomials in one variable over the reals localized at the prime ideal generated by x^2+1. Then A is a commutative, local, principal ideal domain with maximal ideal

$$P = (x^2+1)\mathbb{R}[x]_{(x^2+1)},$$

and the embedding of $\mathbb{R}[x]$ into $\mathbb{C} \simeq \mathbb{R}[x]/(x^2+1)$ given by Lesieur [16, p.116] can be adapted to yield an embedding $i : A \to A/P \simeq \mathbb{C}$. If $\phi : A \to A/P$ denotes the canonical epimorphism, then the ring

$$R = \left\{ \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \middle| a \in A, m \in A/P \right\}$$

with entry by entry addition and multiplication defined by

$$\begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \begin{pmatrix} b & n \\ 0 & b \end{pmatrix} = \begin{pmatrix} ab & i(a)n + m\phi(b) \\ 0 & ab \end{pmatrix}$$

is a right noetherian (but not left noetherian) subdirectly irreducible $\mathbb{Q}$-algebra with right ideals

$$R \supset Q \supset Q^2 \supset \ldots \supset \bigcap_{i=1}^{\infty} Q^i = N \supset 0,$$

where

$$Q = \left\{ \begin{pmatrix} p & m \\ 0 & p \end{pmatrix} \middle| p \in P, m \in A/P \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \middle| m \in A/P \right\}.$$

There is an embedding f of R in the artinian ring $M_2(A/P)$, the map f being given by

$$f\left(\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} i(a) & m \\ 0 & \phi(a) \end{pmatrix}.$$

However, $\mathrm{Ass}(R_R) = Q$, and Q is not a minimal prime, so by Theorem 3.5 no left flat embedding of R into a simple artinian ring exists. The nature of the embedding of A in A/P (cf. [16] for details) shows that as a $\mathbb{Q}$-algebra R has infinite GK-dimension. Note that N is the unique minimal right ideal and that $\ell(N) = N$, so $\mathrm{GK}(A) \geq \mathrm{GK}(N)_R \geq \mathrm{GK}_R(N) = \mathrm{GK}(R/\ell(N))$ for any nonzero right ideal A, hence R is right weakly GK-homogeneous. Thus the example shows that Theorem 3.1 need not be true when $\mathrm{GK}(R)$ is not finite.

4. Artinian Quotient Rings

By Theorem 3.1, a right noetherian right weakly GK-homogeneous k-algebra with finite GK-dimension has a left flat artinian embedding. It turns out that if the algebra is left noetherian as well then this embedding can be obtained without the use of Schofield's Theorem, since in this case there is an artinian quotient ring (see Corollary 4.2 below). This generalizes a result due to Joseph and Small [9], who have shown that any GK-homogeneous homomorphic image of the enveloping algebra of a finite dimensional Lie algebra has an artinian quotient ring. Note, however, that this need no longer be true if the algebra is merely right noetherian, even when it is a PI-algebra. Blair and Small [1] point out that the ring

$$R = \left\{ \begin{pmatrix} f(0) & g(x) \\ 0 & f(x) \end{pmatrix} \middle| f(x), g(x) \in k[x] \right\}$$

is a right noetherian (but not left noetherian) affine PI-algebra with $\mathrm{GK}(R) = 1$. It is irreducible, hence, being right fully bounded, it is right primary and thus right weakly GK-homogeneous by Proposition 3.2. However,

$$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \in \mathcal{C}(N),$$

yet

$$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0,$$

so

$$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \notin \mathcal{C}'(0),$$

and R does not have a right artinian right quotient ring by Small's criterion [19].

Note that a right noetherian, left weakly GK-homogeneous k-algebra R is also right weakly GK-homogeneous. For if A is a nonzero right ideal of R, then it follows from [13, Lemma 5.3] that

$$\mathrm{GK}(R/N) \leq \mathrm{GK}_R(RA) \leq \mathrm{GK}(RA)_R = \mathrm{GK}(A).$$

Thus, in a noetherian algebra one does not have to distinguish between right or left weakly GK-homogeneous. Note, however, that the example above, while right weakly GK-homogeneous, is not left weakly GK-homogeneous. For the prime radical

$$N = \begin{pmatrix} 0 & k[x] \\ 0 & 0 \end{pmatrix}$$

has left annihilator

$$\ell(N) = \left\{ \begin{pmatrix} 0 & g(x) \\ 0 & xf(x) \end{pmatrix} \,\middle|\, f(x), g(x) \in k[x] \right\},$$

so that $R/\ell(N) \simeq k$, whereas $R/N \simeq k[x]$, and consequently

$$\mathrm{GK}_R(N) = \mathrm{GK}(R/\ell(N)) = \mathrm{GK}(k) = 0 < 1 = \mathrm{GK}(k[x]) = \mathrm{GK}(R/N).$$

Note also that in this example

$${}'\mathcal{C}(0) = \mathcal{C}(N) = \left\{ \begin{pmatrix} f(0) & g(x) \\ 0 & f(x) \end{pmatrix} \,\middle|\, f(x) \neq 0 \right\},$$

so the converse of the following result is not true in general. See, however, Theorem 4.3.

4.1 THEOREM: *Let R be a right noetherian k-algebra with finite* GK*-dimension. If R is left weakly GK-homogeneous, then $\mathcal{C}(N) \subseteq {}'\mathcal{C}(0)$.*

Proof: Let $c \in \mathcal{C}(N)$ and assume that $xc = 0$ for some $x \neq 0$, so $x \in \ell(N^{k+1}) \setminus \ell(N^k)$ for some integer $k \geq 0$. Since the elements of $\mathcal{C}(N)$ satisfy the right Ore condition modulo N, the right R/N-module $\ell(N^{k+1})/\ell(N^k)$ contains a nonzero $\mathcal{C}(N)$-torsion submodule $Z/\ell(N^k)$. For any $z \in Z/\ell(N^k)$ there exists $d \in \mathcal{C}(N)$ such that $z(dR+N) = 0$, so

$$\mathrm{GK}(zR) = \mathrm{GK}(R/r(z)) \leq \mathrm{GK}(R/dR+N) < \mathrm{GK}(R/N).$$

Since $Z/\ell(N^k))$ is finitely generated on the right, it follows that $\mathrm{GK}(R/N) > \mathrm{GK}(Z/\ell(N^k))_R$. However, since $Z/\ell(N^k)$ is a bimodule, it follows from [13, Lemma 5.3] that

$$\begin{aligned} \mathrm{GK}(R/N) &> \mathrm{GK}_R(Z/\ell(N^k)) = \mathrm{GK}(R/\ell(Z/\ell(N^k))) \\ &= \mathrm{GK}(R/\ell(ZN^k)) = \mathrm{GK}_R(ZN^k) \geq \mathrm{GK}(R/N), \end{aligned}$$

which gives a contradiction. ∎

4.2 COROLLARY: *A noetherian weakly GK-homogeneous k-algebra R for which $GK(R) < \infty$ has an artinian quotient ring.*

Proof: Apply Theorem 4.1 on both sides to get $\mathcal{C}(N) \subseteq {}'\mathcal{C}(0) \cap \mathcal{C}'(0) = \mathcal{C}(0)$, hence the result by Small's [19] regularity condition. We offer an alternate proof, based on the fact [10, Theorem 2] that a noetherian ring has a artinian quotient ring if and only if its prime middle annihilators are minimal primes. Let $M = \mathrm{Mid}(A, B), AB \neq 0$ be a prime middle annihilator, and assume that M is not a minimal prime, so $\mathrm{GK}(R/M) < \mathrm{GK}(R/N)$. Then, using Lemma 5.3 and Cor. 5.4 of [13],

$$\begin{aligned}\mathrm{GK}(R/N) > \mathrm{GK}(R/M) &= \mathrm{GK}(R/\ell(B/r(A))) = \mathrm{GK}_R(B/r(A))\\ &= \mathrm{GK}(B/r(A))_R = \mathrm{GK}(R(r(B/r(A))) = \mathrm{GK}(R/r(AB))\\ &= \mathrm{GK}(AB)_R \geq \mathrm{GK}(R/N),\end{aligned}$$

a contradiction. ∎

We have seen above that for a noetherian algebra of finite GK-dimension various conditions imply the existence of a left flat artinian embedding. The next result shows that if the algebra has a unique maximal right associated prime, so in particular if it is irreducible, then all these are equivalent.

4.3 THEOREM: *Let R be a noetherian k-algebra with $\mathrm{GK}(R) < \infty$, and assume that $\mathrm{Ass}(R_R)$ has a unique maximal element. Then the following are equivalent.*

(1) *R is right primary.*

(2) *R is weakly GK-homogeneous.*

(3) *R has an artinian quotient ring.*

(4) $\mathcal{C}(N) \subseteq {}'\mathcal{C}(0)$.

(5) $K = \{a \in R \mid \rho(aR) = 0\} = 0$.

(6) *R embeds in a simple artinian ring S such that ${}_RS$ is flat.*

Proof: (1) → (2): This is proved in Proposition 3.2.

(2) → (3): Corollary 4.2.

(3) → (4): This follows since $\mathcal{C}(N) = \mathcal{C}(0)$ by Small [19].

(4) → (5): Since the elements of $\mathcal{C}(N)$ are left regular by hypothesis, $\rho(xR) > 0$ for any nonzero element x, so $K = 0$.

(5) → (6): Theorem 3.5.

(6) → (1): By Theorem 3.5, $\mathrm{Ass}(R_R)$ consists of minimal primes. Since R is assumed to have a unique maximal right associated prime, P say, it follows that $\mathrm{Ass}(R_R) = P$. ∎

Remark: If R is an irreducible noetherian k-algebra of finite GK-dimension, then the following can be added to the above list.

(1′) R is left primary.

(4′) $\mathcal{C}(N) \subseteq \mathcal{C}'(0)$.

(5′) $K' = \{a \in R \mid \rho(Ra) = 0\} = 0$.

(6′) R embeds in a simple artinian ring T such that T_R is flat.

Theorem 3.5 shows that a noetherian k-algebra R that has a left flat artinian embedding admits a rather special primary decomposition. If $\mathrm{GK}(R) < \infty$, then each of the primary components $R/I_j, j = 1, \dots, n$, has an artinian quotient ring $Q(R/I_j)$ by Theorem 4.3. Thus we have an embedding of R in the artinian ring $T = \bigoplus_{j=1}^n Q(R/I_j)$. The following example shows that although each $Q(R/I_j)$ is flat as a left R/I_j-module, ${}_RT$ need not be flat, so the above embedding is not the left flat artinian one we know exists also. The example also shows that a noetherian PI-algebra R with $\mathrm{GK}(R) = 1$ may have a left flat embedding in an artinian ring, but no right flat such embedding need exist.

4.4 Example: Let k be a field, $k[x]$ the commutative polynomial ring in one variable x, and let $M = k[x]/(x)$, viewed as a $(k[x], k)$-bimodule. The ring

$$R = \begin{pmatrix} k[x] & M \\ 0 & k \end{pmatrix}$$

is a noetherian PI-algebra with $\mathrm{GK}(R) = 1$. The minimal prime ideals are

$$P_1 = \begin{pmatrix} k[x] & M \\ 0 & 0 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} 0 & M \\ 0 & k \end{pmatrix}.$$

Setting

$$N = P_1 \cap P_2 = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} (x) & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and } P_3 = \begin{pmatrix} (x) & M \\ 0 & k \end{pmatrix} \supsetneqq P_2,$$

one sees that $P_3 = \ell(N)$, so P_3, being a maximal ideal, is left associated with R. Since P_3 is not a minimal prime, no right flat embedding of R in an artinian ring exists. On the other hand, $P_1 = r(P_2)$, so $\mathrm{Ass}(P_2)_R = P_1$, since P_1 is also maximal. Also, $P_2 = r(I')$ for any right ideal $0 \neq I' \subseteq I$, so $\operatorname{Ass}(I)_R = P_2$. As $I \oplus P_2$ is an essential submodule of R_R, it follows that $\operatorname{Ass}(R_R) = \{P_1, P_2\} = \operatorname{minspec}(R)$, so R has an embedding in an artinian ring S such that ${}_RS$ is flat. Next, note that $I \cap P_2 = 0$, and that both I and P_2 are irreducible ideals. Now $\operatorname{Ass}(R/I)_R = P_1$, so R/I has an artinian quotient ring by Theorem 4.3, in fact $Q(R/I) = R/I$, since

$$R/I \simeq \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$$

is already artinian. As $R/P_2 \simeq k[x]$, $Q(R/P_2) \simeq k(x)$. If $T = Q(R/I) \oplus Q(R/P_2)$ were flat as a left R-module, then $Q(R/I) = R/I$ would have to be a flat left R-module. By a well-known criterion for flatness (see, for example, [14, p.133]), it would follow that $A \cap I = AI$ for each right ideal A of R, so in particular $I = I^2$. Since I is obviously not idempotent, this shows that ${}_RT$ is not flat.

4.5 Example: Let again k be a field, and let

$$R = \begin{pmatrix} k[x] & 0 & M \\ M & k & M \\ 0 & 0 & k \end{pmatrix},$$

where $M = k[x]/(x)$, viewed as the appropriate bimodule or as a ring, depending on its position in the above "matrix" ring. It is easy to see that the minimal primes of R are the ideals

$$P = \begin{pmatrix} k[x] & 0 & M \\ M & 0 & M \\ 0 & 0 & k \end{pmatrix}, \quad Q = \begin{pmatrix} k[x] & 0 & M \\ M & k & M \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & M \\ M & k & M \\ 0 & 0 & k \end{pmatrix}.$$

Furthermore, $\operatorname{Ass}(R_R) = \{Q, S\}$ and $\operatorname{Ass}({}_RR) = \{P, S\}$, so that R has a left flat embedding into an artinian ring and also a right flat such embedding, by Theorem 3.5. However, R does not have an artinian quotient ring. To see this, note that the prime radical is

$$N = P \cap Q \cap S = \begin{pmatrix} 0 & 0 & M \\ M & 0 & M \\ 0 & 0 & 0 \end{pmatrix},$$

and that

$$c = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{C}(N),$$

yet

$$r(c) = \begin{pmatrix} 0 & 0 & M \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{so } c \notin \mathcal{C}'(0),$$

and

$$\ell(c) = \begin{pmatrix} 0 & 0 & 0 \\ M & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{so } c \notin {}'\mathcal{C}(0).$$

5. Prime Middle Annihilators

A noetherian ring that can be embedded in a right artinian ring has only finitely many prime middle annihilators, as has been shown by Dean [3]. Now, in a noetherian ring R the set of all prime ideals that are minimal over some left annihilator ideal is always finite by Cor.2.2 of [7], and it consists of middle annihilators. We proceed to show that for a noetherian k-algebra R with $\mathrm{GK}(R) < \infty$ that has a left flat artinian embedding, the two sets coincide. Recall that the set $\mathcal{S}(0) = \{s \in R \mid Tsx = 0$ for an ideal T implies $Tx = 0\}$ of **strongly regular elements** of the noetherian ring R has been shown to be equal to the set $\bigcap\{\mathcal{C}(P) \mid P$ a prime middle annihilator of $R\}$ (Theorem 2.1 of [6]). On the other hand, if R is a noetherian k-algebra of finite GK-dimension, then

$$\mathcal{C}'(0) = \bigcap\{\mathcal{C}(P) \mid P \text{ a prime minimal over a left annihilator ideal }\}$$

and

$$\mathcal{C}(0) = \bigcap\{\mathcal{C}(P) \,|\, p \text{ a prime ideal minimal over a right or a left annihilator ideal}\}$$

by [11, Théorème 4.4]. Note that in a noetherian ring we always have the inclusions $\mathcal{S}(0) \subseteq \mathcal{C}(0) \subseteq \mathcal{C}'(0)$.

5.1 PROPOSITION: *Let R be a noetherian ring such that*

$$K = \{a \in R \mid \rho(aR) = 0\} = 0.$$

Then $\mathcal{S}(0) = \mathcal{C}(0) = \mathcal{C}'(0)$.

Proof: Let P be a prime middle annihilator, $P = \mathrm{Mid}(A, B)$, A and B ideals with $AB \neq 0$. Let $c \in \mathcal{C}'(0)$, and assume that $c \notin \mathcal{C}(P)$, so $xc \in P$ for some $x \notin P$.

Now $cB \simeq B$ as right R-modules, so $\rho(B/cB) = 0$, hence for each $b \in B$ there exist an element $d \in \mathcal{C}(N)$ such that $bd \in cB$, whence $Axbd \subseteq AxcB \subseteq APB = 0$. Consequently, $\rho(AxB) = 0$, so $AxB \subseteq K = 0$, whence $x \in \text{Mid}(A, B) = P$. This contradiction shows that $\mathcal{C}'(0) \subseteq \mathcal{C}(P)$ for any prime middle annihilator P, so $\mathcal{C}'(0) \subseteq \mathcal{S}(0)$, which, by the remark above, is all we had to prove. ∎

We need the following unpublished result due to T. H. Lenagan, presented during the conference "Noetherian rings and rings with polynomial identity", held at the University of Durham, July 23–31, 1979.

5.2 Lemma: *Let R be a noetherian ring with ideals I_1 and I_2 such that $I_1 \cap I_2 = 0$. If P is a prime middle annihilator of R, then $I_1 \subseteq P$ or $I_2 \subseteq P$, and P/I_1 is a prime middle annihilator of R/I_1 or P/I_2 is a prime middle annihilator of R/I_2.*

Proof: Let $P = \text{Mid}(A, B), AB \neq 0$. Assume that $AB \subseteq I_1$ but $AB \not\subseteq I_2$. Then $A(P + I_2)B = AI_2B \subseteq AB \cap I_2 = 0$, whence $I_2 \subseteq P$ and

$$P/I_2 \subseteq \text{Mid}(A + I_2/I_2, B + I_2/I_2) = M/I_2,$$

say. Since $AMB \subseteq I_2 \cap AB = 0$, $M \subseteq \text{Mid}(A, B) = P$, so $M = P$, and P/I_2 is a middle annihilator in R/I_2. If $AB \not\subseteq I_j$ for $j = 1, 2$, set

$$N_j/I_j = \text{Mid}(A + I_j/I_j, B + I_j/I_j).$$

Then $A(N_1 \cap N_2)B \subseteq I_1 \cap I_2 = 0$, so $N_1 \cap N_2 \subseteq P$, hence $N_1 \subseteq P$ or $N_2 \subseteq P$. Since clearly $P \subseteq N_1$ and $P \subseteq N_2$, it follows that $P = N_1$ or $P = N_2$. ∎

5.3 Theorem: *Let R be a noetherian k-algebra with $\text{GK}(R) < \infty$ that embeds in an artinian ring S such that ${}_RS$ is flat. Then each prime middle annihilator of R is a prime minimal over a left annihilator ideal.*

Proof: By Theorem 3.5, there exist irreducible right P_j-primary ideals $I_1, I_2, \ldots, I_n$ such that

$$I_1 \cap I_2 \cap \ldots \cap I_n = 0 \quad \text{and} \quad \{P_1, \ldots, P_n\} \subseteq \text{minspec}(R).$$

Furthermore, the ideals I_j can be chosen such that whenever $X_j \supsetneqq I_j$ for some j, then $X_j \cap \bigcap_{i \neq j} I_i \neq 0$. Now let P be a prime middle annihilator. Assume first that P is a maximal middle annihilator. If P were not minimal over some left annihilator ideal, then we would have that $P \cap \mathcal{C}(Q) \neq \emptyset$ for each prime Q in the set $\mathcal{M}$ of primes that are minimal over a left annihilator ideal. Since the set $\mathcal{M}$ is finite by [7, Cor. 2.2], this would imply that $P \cap \bigcap\{\mathcal{C}(Q) \mid Q \in \mathcal{M}\} \neq \emptyset$ by [20, Prop. 2.4]. Since $\bigcap\{\mathcal{C}(Q) \mid Q \in \mathcal{M}\} = \mathcal{C}'(0)$ by [11, Théorème 4.4], and since $\mathcal{S}(0) = \mathcal{C}'(0)$ by Proposition 5.1, this gives a contradiction. Let now P be a non-maximal prime middle annihilator, and let Q_1 be a prime middle annihilator, $Q_1 \supsetneqq P$. By Lemma 5.2, Q_1 is a prime middle annihilator over some of the ideals I_j, say over $I_1, \ldots, I_{n(1)}$. By Theorem 4.3, each R/I_j has an artinian quotient ring, so Q_1 is a minimal prime over each I_j by [10, Theorem 2], hence Q_1 is minimal over $\bigcap\{I_j \mid 1 \leq j \leq n(1)\}$. It follows that P does not contain $\bigcap\{I_j \mid 1 \leq j \leq n(1)\}$, so by Lemma 5.2, P must be a prime middle annihilator over $\bigcap\{I_j \mid n(1) < j \leq n\}$. If P is not a maximal middle annihilator over $\bigcap\{I_j \mid n(1) < j \leq n\}$, then we continue in this fashion, until eventually we get that P does not contain $D' = I_1 \cap \ldots \cap I_m$ for some $1 \leq m < n$, and that P is a maximal middle annihilator over $D = I_{m+1} \cap \ldots \cap I_n$. By Theorem 3.5, R/D satisfies the same hypotheses as R, so by the first part of this proof, P/D is minimal over some left annihilator ideal in R/D, say over $Y/D = \ell_{R/D}(X/D)$. Since $YX \subseteq D$, we have that $(Y \cap D')X \subseteq YX \cap D' \subseteq D \cap D' = 0$, so $Y \cap D' \subseteq \ell_R(X) \subseteq \ell_R(X/D) = Y \subseteq P$. Since P is a prime minimal over Y, and since $D' \not\subseteq P$, P is minimal over $Y \cap D'$, hence also minimal over $\ell_R(X)$, thus proving the result. ∎

5.4 COROLLARY: *Let R be a noetherian k-algebra of finite GK-dimension that has a left flat artinian embedding. Let*

$$0 = B_0 \subset B_1 \subset \ldots \subset B_{i-1} \subset B_i \subset \ldots \subset B_n = R$$

be any full series of left annihilator ideals, that is, no left annihilator ideal exists between B_{i-1} and B_i, and let $P_i = r(B_i/B_{i-1})$. Then the set $\{P_1, \cdots, P_n\}$ is the set of all prime middle annihilators of R.

Proof: This follows immediately from Theorem 5.3 and [7, Theorem 2.1]. ∎

The above characterization of the set of prime middle annihilators is by no means restricted to noetherian algebras of finite GK-dimension that have a left flat embedding in an artinian ring. In fact, the ring U of [5], shown to have no embedding in an artinian ring whatsoever, has only two prime middle annihilators, both of which are actually left annihilator ideals. For this ring U, a homomorphic image of $U(\mathfrak{sl}(2,\mathbb{C}))$, has precisely two proper nonzero ideals, P and Q with $Q^2 = Q$, $QP = PQ = 0$, and $Q \supset P$. Obviously, $Q = \ell(P)$ and $P = \ell(Q)$.

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